

PRYM VARIETIES OF ÉTALE COVERS OF HYPERELLIPTIC CURVES

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ABSTRACT. It is well known that the Prym variety of an étale cyclic covering of a hyperelliptic curve is isogenous to the product of two Jacobians. Moreover, if the degree of the covering is odd or congruent to 2 mod 4, then the canonical isogeny is an isomorphism. We compute the degree of this isogeny in the remaining cases and show that only in the case of coverings of degree 4 it is an isomorphism.

1. INTRODUCTION

Let H denote a hyperelliptic curve of genus $g \geq 2$ and $f : X \rightarrow H$ an étale cyclic covering of degree $n \geq 2$. Let σ denote the automorphism of X defining f . It is well known that the hyperelliptic involution of H lifts to an involution τ on X . Then σ and τ generate the dihedral group D_n of order $2n$. The Prym variety $P(f)$ of f is defined as the connected component containing 0 of the kernel of the norm map $\mathrm{Nm} f : JX \rightarrow JH$ of f . For any element $\alpha \in D_n$ we denote by X_α the quotient of X by the subgroup generated by α . The Jacobians JX_τ and $JX_{\tau\sigma}$ are abelian subvarieties of the Prym variety $P(f)$ so the the addition map

$$a : JX_\tau \times JX_{\tau\sigma} \rightarrow P(f)$$

is well defined. Mumford showed in [3] that for $n = 2$ the map a is an isomorphism. J. Ries proved the same for any odd prime degree n ([6]). The second author generalized this statement largely to show that a is an isomorphism for any odd number and, more important, for any even $n \equiv 2 \pmod{4}$ ([4]).

It is an obvious question whether this is true for any degree n . In fact, using the action of the group D_n on $P(f)$ and a little representation theory, it is not difficult to see that a is an isogeny. For more precise results on the decomposition of $P(f)$ up to isogeny see [2]. It is the aim of this note to compute the degree of the isogeny a . Our main result is

Theorem 4.1 *Let $f : X \rightarrow H$ be as above with $n = 2^r m$, $r \geq 2$ and m odd. Then a is an isogeny of degree*

$$\deg a = 2^{[(2^r - r - 1)m - (r - 1)](g - 1)}.$$

So a is an isomorphism for odd n , for $n \equiv 2 \pmod{4}$, and for $n = 4$. The proof proceeds by induction on the exponent r , the beginning of the induction being Ortega's theorem in [4].

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2. PRELIMINARIES

Let H be a smooth hyperelliptic curve of genus g with hyperelliptic covering $\pi : H \rightarrow \mathbb{P}^1$ and $f : X \rightarrow H$ be a cyclic étale covering of degree $n \geq 2$. So X is of genus $g_X = n(g-1)+1$ and the Prym variety $P := P(f)$ of f is an abelian variety of dimension

$$(2.1) \quad \dim P = (n-1)(g-1).$$

The canonical polarization of JX induces a polarization on P of type

$$(\underbrace{1, \dots, 1}_{(n-2)(g-1)}, \underbrace{n, \dots, n}_{g-1}).$$

The hyperelliptic involution of H lifts to an involution τ on X which together with the automorphism σ defined by the covering f generate the dihedral group

$$D_n := \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle.$$

The automorphism σ induces an automorphism of the same order n of P compatible with the polarization, which we denote by the same letter. Each eigenvalue $\zeta_n^i, i = 1, \dots, n-1$ (with ζ_n a fixed primitive n -th root of unity) of the induced map on the tangent space T_0P occurs with multiplicity $g-1$.

In the whole paper we write

$$n = 2^r m$$

with $r \geq 0$ and m odd.

In any case the group D_n admits n involutions, namely $\tau\sigma^\nu$ for $\nu = 0, \dots, n-1$. For odd n , these are all the involutions. For even n , there is one more, namely $\sigma^{n/2}$. For odd n all involutions are conjugate to τ and for even n there are 3 conjugacy classes. They are represented by

$$\tau, \tau\sigma^m \quad \text{and} \quad \sigma^{n/2}.$$

For any subgroup $S \subset D_n$ and for any element $\alpha \in D_n$ we denote by

$$X_S := X/S \quad \text{and} \quad X_\alpha := X/\langle \alpha \rangle$$

the corresponding quotients.

Consider the following diagram (for odd n only the left hand side of the diagram, since in this case $m = n$, so both sides are the same).

$$\begin{array}{ccccc} & & X & & \\ & \swarrow 2:1 & \downarrow f \quad n:1 & \searrow 2:1 & \\ X_\tau & & H & & X_{\tau\sigma^m} \\ & \swarrow n:1 & \downarrow \pi \quad 2:1 & \searrow n:1 & \\ & & \mathbb{P}^1 & & \end{array}$$

Let W denote the set of $2g+2$ branch points of the hyperelliptic covering π . Then denote for arbitrary n ,

$$s_0 := |\{x \in W \mid (\pi f)^{-1}(x) \text{ contains a fixed point of } \tau\}|$$

and

$$s_1 := |\{x \in W \mid (\pi f)^{-1}(x) \text{ contains a fixed point of } \tau\sigma^m\}|.$$

According to [4, Proposition 2.4] the Jacobians JX_τ and $JX_{\tau\sigma^m}$ are contained in the Prym variety P . With these notations the following theorem is proved in [4].

Theorem 2.1. (a) For odd n the map

$$\psi : (JX_\tau)^2 \rightarrow P, \quad (x, y) \mapsto x + \sigma(y)$$

is an isomorphism.

(b) For $n = 2m \equiv 2 \pmod{4}$ the map

$$\psi : JX_\tau \times JX_{\tau\sigma^m} \rightarrow P, \quad (x, y) \mapsto x + y$$

is an isomorphism. Moreover,

$$g(X_\tau) = m(g-1) + 1 - \frac{s_0}{2} \quad \text{and} \quad g(X_{\tau\sigma^m}) = m(g-1) + 1 - \frac{s_1}{2}.$$

In particular s_0 and s_1 are even.

It is the aim of this paper to study the map ψ in the remaining cases $n = 2^r m$ with $r \geq 2$. So in the sequel we assume $r \geq 2$. We first need some preliminaries.

There are 2 non-conjugate Kleinian subgroups of D_n , namely

$$K_\tau = \{1, \sigma^{n/2}, \tau, \tau\sigma^{n/2}\} \quad \text{and} \quad K_{\tau\sigma^m} = \{1, \sigma^{n/2}, \tau\sigma^m, \tau\sigma^{m+n/2}\}.$$

Moreover, consider the dihedral subgroups of order 8,

$$T_\tau = \langle \tau, \sigma^{n/4} \rangle \quad \text{and} \quad T_{\tau\sigma^m} = \langle \tau\sigma^m, \sigma^{n/4} \rangle.$$

Note that for $r \geq 3$ the groups T_τ and $T_{\tau\sigma^m}$ are non-conjugate, whereas

$$(2.2) \quad T_\tau = T_{\tau\sigma^m} \quad \text{for} \quad r = 2,$$

since then $\frac{n}{4} = m$ and $\langle \tau, \sigma^m \rangle = \langle \tau\sigma^m, \sigma^m \rangle$. In any case we have the following commutative diagram

(2.3)

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow a_{\tau\sigma^{n/2}} & \downarrow f_1 \quad 2:1 & \searrow a_{\tau\sigma^m} & \\
 X_{\tau\sigma^{n/2}} & & X_{\sigma^{n/2}} & & X_{\tau\sigma^m} \\
 \downarrow b_{\tau\sigma^{n/2}} & \swarrow c_{\tau\sigma^{n/2}} & \downarrow f_2 \quad 2:1 & \searrow c_{\tau\sigma^m} & \downarrow b_{\tau\sigma^m} \\
 X_{K_\tau} & & X_{\sigma^{n/4}} & & X_{K_{\tau\sigma^m}} \\
 \downarrow d_{\tau\sigma^{n/2}} & \swarrow e_{\tau\sigma^{n/2}} & \downarrow f_3 \quad \frac{n}{4}:1 & \searrow e_{\tau\sigma^m} & \downarrow d_{\tau\sigma^m} \\
 X_{T_\tau} & & X_\sigma = H & & X_{T_{\tau\sigma^m}} \\
 \swarrow \frac{n}{4}:1 & & \downarrow \pi & & \swarrow \frac{n}{4}:1 \\
 & & \mathbb{P}^1 & &
 \end{array}$$

In the sequel we use the following notation: if an involution of the group D_n induces an involution on a curve of the diagram, we denote the induced involution by the same letter. In order to compute the genera of the curves in the diagram, we need the following lemma.

Lemma 2.2. *Suppose that the dihedral group $D_n = \langle \sigma, \tau \rangle$ of order $2n$ with $n \geq 3$ acts on a finite set S of n elements such that the subgroup $\langle \sigma \rangle$ acts transitively on S .*

- (a) *If n is odd, τ admits exactly one fixed point,*
- (b) *for even n , either τ acts fixed-point free or admits exactly 2 fixed points.*
- (c) *for even n , exactly one of the involutions τ and $\tau\sigma$ admits a fixed point.*

Proof. Let $S = \{x_1, \dots, x_n\}$. We may enumerate the x_i in such a way that $\sigma(x_i) = x_{i+1}$ for $i = 1, \dots, n$ where $x_{n+1} = x_1$. If n is odd, then clearly τ admits a fixed point. So in any case we may assume that x_1 is a fixed point of τ . Then we have inductively for $i = 1, \dots, \lfloor \frac{n+3}{2} \rfloor$,

$$(2.4) \quad \tau(x_i) = x_{n+2-i}.$$

In fact, the induction step is $\tau(x_i) = \tau\sigma(x_{i-1}) = \sigma^{-1}\tau(x_{i-1}) = \sigma^{-1}(x_{n-i+3}) = x_{n-i+2}$. Hence for odd n the involution τ admits no further fixed point and for even n τ admits exactly one additional fixed point, namely $x_{\frac{n+2}{2}}$. This gives (a) and (b).

(c): Suppose n is even and τ admits a fixed point, say x_1 . Hence we have (2.4) for all i . This implies

$$\tau\sigma(x_i) = \tau(x_{i+1}) = x_{n+1-i}.$$

and $\tau\sigma$ acts fixed point free. Conversely, suppose τ acts fixed point free. Suppose that $\tau(x_1) = x_i$ for some $i \geq 2$. Then $\sigma^{1-i}\tau(x_1) = x_1$. So $\sigma^{1-i}\tau$ admits a fixed point and thus cannot be equivalent to τ . Hence $\tau\sigma$ is equivalent to $\sigma^{1-i}\tau$ and admits a fixed point. \square

Lemma 2.3. *Suppose $n = 2^r m$ with m odd and $r \geq 2$. Then*

(i)

$$s_0 + s_1 = 2g + 2 \quad \text{with} \quad s_0, s_1 \geq 2 \text{ even};$$

(ii) *for $r = 2$, $X_{\sigma^{n/4}} \rightarrow X_{T_\tau}$ and $X_{\sigma^{n/4}} \rightarrow X_{T_{\tau\sigma^m}}$ are ramified exactly at $2g + 2$ points.*

(iii) *$X \rightarrow X_\tau$ and $X_{\sigma^{n/2}} \rightarrow X_{K_\tau}$ as well as $X_{\sigma^{n/4}} \rightarrow X_{T_\tau}$, if $r \geq 3$, are ramified exactly at $2s_0$ points. $X \rightarrow X_{\tau\sigma^m}$ and $X_{\sigma^{n/2}} \rightarrow X_{K_{\tau\sigma^m}}$ as well as $X_{\sigma^{n/4}} \rightarrow X_{T_{\tau\sigma^m}}$, if $r \geq 3$, are ramified exactly at $2s_1$ points.*

Proof. The fixed points of τ and $\tau\sigma^m$ lie over the $2g+2$ Weierstrass points of H . Moreover, according to Lemma 2.2. over each Weierstrass point of H exactly one of τ and $\tau\sigma^m$ admits a fixed point. This gives the first assertion of (i). The evenness of s_0 and s_1 follows from the Hurwitz formula. Now $s_0 = 0$ means that τ acts fixed-point free. Since also σ acts fixed-point free, so does $\tau\sigma^m$ which means $s_1 = 0$. But this contradicts the equation $s_0 + s_1 = 2g + 2$. Hence $s_0, s_1 \geq 2$.

If x is a Weierstrass point of H and τ admits a fixed point over x , then D_n acts on the fibre $f^{-1}(x)$. Similarly, the group $D_{n/2} = \langle \sigma^{n/2}, \tau \rangle$ acts on the fibre $(f_3 \circ f_2)^{-1}(x)$ and the group $D_{n/4} = \langle \sigma^{n/4}, \tau \rangle$ acts on the fibre $f_3^{-1}(x)$. Hence Lemma 2.2 implies (ii), since in these cases the order of the fibre is even, and (iii), since in this case the order of the fibre is odd. \square

By checking the ramification of the maps in diagram (2.3) we immediately get from Lemma 2.3 the following corollaries.

Corollary 2.4. *All vertical left and right hand maps are ramified.*

Corollary 2.5. *If $n = 2^r m$ with m odd and $r \geq 2$, then*

$$g(X) = n(g-1) + 1 \quad g(X_{\sigma^{n/2}}) = \frac{n}{2}(g-1) + 1 \quad g(X_{\sigma^{n/4}}) = \frac{n}{4}(g-1) + 1;$$

$$g(X_{\tau\sigma^{n/2}}) = \frac{n}{2}(g-1) + 1 - \frac{s_0}{2} \quad g(X_{\tau\sigma^m}) = \frac{n}{2}(g-1) + 1 - \frac{s_1}{2};$$

$$g(X_{K_\tau}) = \frac{n}{4}(g-1) + 1 - \frac{s_0}{2} \quad g(X_{K_{\tau\sigma^m}}) = \frac{n}{4}(g-1) + 1 - \frac{s_1}{2};$$

and for $r \geq 3$,

$$g(X_{T_\tau}) = \frac{n}{8}(g-1) + 1 - \frac{s_0}{2} \quad g(X_{\tau\sigma^m}) = \frac{n}{8}(g-1) + 1 - \frac{s_1}{2}.$$

For $r = 2$,

$$g(X_{T_\tau}) = g(X_{\tau\sigma^m}) = \frac{1}{2}(m-1)(g-1).$$

Proof. All assertions follow from the Hurwitz formula. For the first line of assertions we use the fact that f is étale. For the other formulas we use Lemma 2.3(ii) and (iii). \square

The following lemma is well known. In fact, it is an easy consequence of [1, Proposition 11.4.3] and [1, Corollary 12.1.4].

Lemma 2.6. *Let $g : Y \rightarrow Z$ be a covering of smooth projective curves of degree $d \geq 2$. The addition map*

$$g^* JZ \times P(Y/Z) \rightarrow JY$$

is an isogeny of degree

$$|g^* JZ \cap P(Y/Z)| = \frac{|JZ[d]|}{|\ker g^*|^2}.$$

We need a result on curves with an action of the Klein group. Let Y be a curve with an action of the group

$$V_4 = \langle r, s \mid r^2 = s^2 = (rs)^2 = 1 \rangle.$$

Then we have the following diagram

$$(2.5) \quad \begin{array}{ccccc} & & Y & & \\ & \swarrow a_s & \downarrow a_r & \searrow a_{rs} & \\ Y_s & & Y_r & & Y_{rs} \\ & \searrow & \downarrow & \swarrow & \\ & & Z & & \end{array}$$

with $Y_v := Y/\langle v \rangle$ for any $v \in V_4$ and $Z = Y/V_4$. The following theorem is a special case of [5, Theorem 3.2].

Proposition 2.7. *Suppose a_r is étale, that a_s respectively a_{rs} are ramified at $2\alpha_s > 0$, respectively $2\alpha_{rs} > 0$ points and Z is of genus $g(Z)$. Then $P(Y_s/Z)$ and $P(Y_{rs}/Z)$ are subvarieties of $P(Y/Y_r)$ and the addition map*

$$\phi_r : P(Y_s/Z) \times P(Y_{rs}/Z) \rightarrow P(Y/Y_r)$$

is an isogeny of degree $2^{2g(Z)}$.

3. A DEGREE COMPUTATION

As above, let $n = 2^r m$ with m odd and $r \geq 2$. Again we consider a curve X with action of the dihedral group $D_n := \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle$. With the notation as in Section 2 we have diagram (2.3) and $s_0, s_1 \geq 2$. Then, apart from f_1, f_2 and f_3 , all the maps in diagram (2.3) are ramified. So the pullbacks of the corresponding Jacobians are embeddings. Recall that $P(f)$ denotes the Prym variety of the covering f .

We consider the isogenies

$$h := \text{Nm } a_{\tau\sigma^m} \circ a_{\tau\sigma^{n/2}}^* : P(b_{\tau\sigma^{n/2}}) \longrightarrow P(b_{\tau\sigma^m}).$$

and

$$h' := \text{Nm } a_{\tau\sigma^{n/2}} \circ a_{\tau\sigma^m}^* : P(b_{\tau\sigma^m}) \longrightarrow P(b_{\tau\sigma^{n/2}}).$$

Let

$$A := a_{\tau\sigma^{n/2}}^*(P(b_{\tau\sigma^{n/2}})) \quad \text{and} \quad B := a_{\tau\sigma^m}^*(P(b_{\tau\sigma^m}))$$

be subvarieties of JX . Now τ (respectively $\sigma^{n/2}$) induces an involution on $X_{\tau\sigma^{n/2}}$ (respectively $X_{\tau\sigma^m}$), which we denote by the same letter. Thus the Prym variety $P(b_{\tau\sigma^{n/2}})$ is $\text{Ker}(1 + \tau)^0$ and $P(b_{\tau\sigma^m}) = \text{Ker}(1 + \sigma^{n/2})^0$. Hence we have (for example by [5, Corollary 2.7]),

$$A = \{z \in JX^{\langle \tau\sigma^{n/2} \rangle} \mid z + \tau z = 0\}^0, \quad B = \{w \in JX^{\langle \tau\sigma^m \rangle} \mid w + \sigma^{n/2} w = 0\}^0.$$

Moreover, as in [5], there is a commutative diagram:

$$(3.1) \quad \begin{array}{ccccc} & & A & \xrightarrow{1+\tau\sigma^m} & B & \xrightarrow{1+\tau\sigma^{n/2}} & A \\ & \nearrow a_{\tau\sigma^{n/2}}^* & \searrow \text{Nm } a_{\tau\sigma^m} & \nearrow a_{\tau\sigma^m}^* & \searrow \text{Nm } a_{\tau\sigma^{n/2}} & \nearrow a_{\tau\sigma^{n/2}}^* & \\ P(b_{\tau\sigma^{n/2}}) & \xrightarrow{h} & P(b_{\tau\sigma^m}) & \xrightarrow{h'} & P(b_{\tau\sigma^{n/2}}) & \end{array}$$

Lemma 3.1. *For any $n = 2^r m$ with m odd and $r \geq 2$ we have*

$$|\text{Ker } h| = |\text{Ker}(1 + \tau\sigma^m)|_A|$$

and

$$\text{Ker}(1 + \tau\sigma^m)_A = (JX[2])^{\langle \tau, \sigma^m \rangle}.$$

Proof. The first assertion follows from diagram (3.1), since $a_{\tau\sigma^{n/2}}^* : P(b_{\tau\sigma^{n/2}}) \rightarrow A$ and $a_{\tau\sigma^m}^* : P(b_{\tau\sigma^m}) \rightarrow B$ are isomorphisms. For the last assertion note that $z \in \text{Ker}(1 + \tau\sigma^m)|_A$ if and only if

$$\tau\sigma^{n/2}z = z, \quad \tau z = -z \quad \tau\sigma^m(z) = -z,$$

which implies that $\sigma^m z = z$. So

$$z = \tau\sigma^{n/2}z = \tau\sigma^{2^{r-1}m}z = \tau(\sigma^m)^{2^{r-1}}(z) = \tau z = -z,$$

then $z \in A[2]$.

Therefore $z \in \text{Ker}(1 + \tau\sigma^m)|_A$ if and only if $z \in JX[2]$ such that $\tau z = z$ and $\sigma^m z = z$ which was to be shown. \square

The following proposition is a generalization of a special case of [5, Theorem 4.1,(ii)].

Proposition 3.2. *For every $n = 2^r m$ with m odd and $r \geq 2$, we have*

$$\deg h = 2^{(m-1)(g-1)+s_1-2}.$$

Proof. The proof is by induction on the exponent $r \geq 2$. Suppose first $r = 2$, i.e. $n = 4m$. Consider the curve X with the action of the dihedral subgroup

$$D_4 := \langle \sigma^m, \tau \rangle \subset D_n.$$

It has 2 non-conjugate Kleinian subgroups, namely $K_\tau = \langle \sigma^{2m}, \tau \rangle$ and $K_{\tau\sigma^m} = \langle \sigma^{2m}, \tau\sigma^m \rangle$. Note that by (2.2), $T_\tau = T_{\tau\sigma^m} = \langle \sigma^m, \tau \rangle$. Then according to [5, Theorem 4.1,(ii)]

$$|\text{Ker } h| = 2^{2g(X_{T_\tau})-2+s_1}.$$

So Corollary 2.5 give the proposition in this case.

Suppose now $r \geq 3$ and the proposition holds for $r - 1$. Let X be a curve with an action of D_n with $X/\langle \sigma \rangle = H$, so that we have the diagram (2.3). Then the subgroup $D_{\frac{n}{2}} = \langle \sigma^2, \tau \rangle$ of index 2 acts of the curve $X_{n/2}$, so that we can apply the inductive hypothesis. This gives that the map

$$h_{n/2} := \text{Nm } c_{\tau\sigma^m} \circ c_{\tau\sigma^{n/2}}^* : P(d_{\tau\sigma^{n/2}}) \longrightarrow P(d_{\tau\sigma^m})$$

is an isogeny of degree $2^{(m-1)(g-1)-2+s_1}$.

Hence it suffices to show that

$$\text{Ker } h = b_{\tau\sigma^{n/2}}^*(\text{Ker } h_{n/2}).$$

This implies the proposition, since the map $b_{\tau\sigma^{n/2}}^*$ is injective.

Now Lemma 3.1 applied to the induction hypothesis, i.e. to $h_{\frac{n}{2}}$ gives

$$|\text{Ker } h_{\frac{n}{2}}| = |(JX_{\tau\sigma^{n/2}}[2])^{\langle \tau, \sigma^m \rangle}| = |a_{\tau\sigma^{n/2}}^*(JX_{\tau\sigma^{n/2}}[2])^{\langle \tau, \sigma^m \rangle}|.$$

But

$$\begin{aligned} a_{\tau\sigma^{n/2}}^*(JX_{\tau\sigma^{n/2}}[2])^{\langle \tau, \sigma^m \rangle} &= \{z \in JX[2] \mid \tau\sigma^{n/2}z = z, \tau z = z, \sigma^m z = z\} \\ &= \{z \in JX[2] \mid \tau z = z, \sigma^m z = z\}, \end{aligned}$$

since the equation $\tau\sigma^{n/2}z = z$ is a consequence of the last 2 equations. This gives

$$|\text{Ker } h| = |\text{Ker } h_{\frac{n}{2}}|$$

which completes the proof of Proposition 3.2. \square

4. DECOMPOSITION FOR $n = 2^r m$, $r \geq 2$ WITH m ODD

Now let the notation be as in Section 1 with $n = 2^r m$, $r \geq 2$ and m odd. Let $f : X \rightarrow H$ be a cyclic étale covering of degree n of a hyperelliptic curve H . The main result of the paper is the following theorem.

Theorem 4.1. *Let n and $f : X \rightarrow H$ be as above. Then JX_τ and $JX_{\tau\sigma^m}$ are abelian subvarieties of the Prym variety $P(f)$ and the addition map*

$$a : JX_\tau \times JX_{\tau\sigma^m} \rightarrow P(f)$$

is an isogeny of degree $2^{[(2^r - r - 1)m - (r - 1)](g - 1)}$.

The proof is by induction on r . Since the proofs for $r = 2$ and for the inductive step in case $r \geq 3$ are almost the same, we give them simultaneously. The difference is only that for $r = 2$ we use Theorem 2.1 instead of the induction hypothesis.

So in the whole of this section we assume that for $r \geq 3$, Theorem 4.1 is true for $r - 1$, i.e. for covering of degree $2^{r-1}m$ for all m . Let $r \geq 2$ and $f : X \rightarrow H$ be an étale covering of degree $n = 2^r m$ with odd $m \geq 1$. We use the notation of diagram (2.3). In addition let $b_\tau : X_\tau \rightarrow X_{K_\tau}$ denote the canonical projection.

Proposition 4.2. *The varieties JX_{K_τ} , $JX_{K_{\tau\sigma^m}}$, $P(b_\tau)$ and $P(b_{\tau\sigma^{n/2}})$ are abelian subvarieties of JX and the addition map*

$$\tilde{\phi}_n : f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^{n/2}}) \rightarrow JX$$

is an isogeny of degree

$$\deg \tilde{\phi}_n = m^{2g-2} \cdot 2^{[(2^{r+1}-r)m+r](g-1)+2-s_0}.$$

Proof. All maps in diagram (2.3) are ramified apart from f_1, f_2 and f_3 , which gives the first assertion. The dihedral group $D_{n/2} = \langle \sigma^2, \tau \rangle$ acts on the curve $X_{\sigma^{n/2}}$.

If $r = 2$, we can apply Theorem 2.1 to get that the canonical map

$$\alpha : JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \rightarrow P(f_3 \circ f_2)$$

is an isomorphism. For $r \geq 3$ we can apply the induction hypothesis, which gives that α is an isogeny of degree $2^{[(2^{r-1}-r)m-(r-2)](g-1)}$. Since this number is equal to 1 for $r = 2$, this is valid for all $r \geq 2$.

Now the addition map $\alpha_1 : (f_3 \circ f_2)^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \rightarrow JX_{\sigma^{n/2}}$ factorizes as

$$\begin{array}{ccc} (f_3 \circ f_2)^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} & \xrightarrow{\alpha_1} & JX_{\sigma^{n/2}} \\ & \searrow \text{id} \times \alpha & \nearrow \psi \\ & (f_3 \circ f_2)^* JH \times P(f_3 \circ f_2) & \end{array}$$

where ψ is the addition map. So Lemma 2.6 implies that

$$\deg \alpha_1 = \deg \alpha \cdot \deg \psi = 2^{[(2^{r-1}-r)m-(r-2)](g-1)} \cdot (2^{r-1}m)^{2g-2} = m^{2g-2} \cdot 2^{[(2^{r-1}-r)m+r](g-1)}.$$

Clearly α_1 and its pullback via f_1^* are of the same degree. Moreover, considering X with the action of the Klein group $\langle \sigma^{n/2}, \tau \rangle$, we have the diagram

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow a_\tau & \downarrow f_1 & \searrow a_{\tau\sigma^{n/2}} & \\
 X_\tau & & X_{\sigma^{n/2}} & & X_{\tau\sigma^{n/2}} \\
 & \searrow b_\tau & \downarrow c_{\tau\sigma^{n/2}} & \swarrow b_{\tau\sigma^{n/2}} & \\
 & & X_{K_\tau} & &
 \end{array}$$

Then Proposition 2.7 gives that the addition map

$$\alpha_2 : P(b_\tau) \times P(b_{\tau\sigma^{n/2}}) \rightarrow P(f_1)$$

is an isogeny of degree $2^{2g(X_{K_\tau})} = 2^{2^{r-1}m(g-1)+2-s_0}$.

Now note that the map $\tilde{\phi}_n$ factorizes as

$$\begin{array}{ccc}
 [f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}}] \times [P(b_\tau) \times P(b_{\tau\sigma^{n/2}})] & \xrightarrow{\tilde{\phi}_n} & JX \\
 \downarrow f_1^* \alpha_1 \times \alpha_2 & \nearrow \alpha_3 & \\
 f_1^* JX_{\sigma^{n/2}} \times P(f_1) & &
 \end{array}$$

By Lemma 2.6 the addition map α_3 is an isogeny of degree $2^{2g(X_{\sigma^{n/2}})-2} = 2^{2^r m(g-1)}$, therefore the map $\tilde{\phi}_n$ is an isogeny of degree

$$\begin{aligned}
 \deg \tilde{\phi}_n &= \deg f_1^* \alpha_1 \cdot \deg \alpha_2 \cdot \deg \alpha_3 \\
 &= m^{2g-2} \cdot 2^{[(2^{r-1}-r)m+r](g-1)} \cdot 2^{2^{r-1}m(g-1)+2-s_0} \cdot 2^{2^r m(g-1)} \\
 &= m^{2g-2} \cdot 2^{[(2^{r+1}-r)m+r](g-1)+2-s_0}.
 \end{aligned}$$

□

Corollary 4.3. *The canonical map*

$$\phi_n : f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) \rightarrow JX$$

is an isogeny of degree

$$\deg \phi_n = m^{2g-2} 2^{[(2^{r+1}-r-1)m+r-1](g-1)}.$$

Proof. According to Proposition 3.2 the canonical map

$$h : P(b_{\tau\sigma^{n/2}}) \rightarrow P(b_{\tau\sigma^m})$$

is an isogeny of degree $2^{(m-1)(g-1)-2+s_1}$.

Now with the definition of the map h one checks that the following diagram commutes

$$\begin{array}{ccc}
 [f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau)] \times P(b_{\tau\sigma^{n/2}}) & \xrightarrow{\tilde{\phi}_n} & JX \\
 \downarrow id \times h & \nearrow \phi_n & \\
 [f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau)] \times P(b_{\tau\sigma^m}) & &
 \end{array}$$

So Propositions 4.2 and 3.2 imply that ϕ_n is an isogeny of degree

$$\begin{aligned} \deg \phi_n &= \frac{\deg \tilde{\phi}_n}{\deg h} \\ &= \frac{m^{2g-2} \cdot 2^{[(2^{r+1}-r)m+r](g-1)+2-s_0}}{2^{(m-1)(g-1)-2+s_1}} = m^{2g-2} \cdot 2^{[(2^{r+1}-r-1)m+r-1](g-1)} \end{aligned}$$

where we used again that $s_0 + s_1 = 2g + 2$. \square

Corollary 4.4. *The canonical map*

$$\psi_n : JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) \rightarrow P(f)$$

is an isogeny of degree $2^{[(2^{r+1}-r-1)m-(r+1)](g-1)}$.

Proof. Clearly the addition maps the source of ψ_n into $P(f)$ and the following diagram is commutative

$$\begin{array}{ccc} f^*JH \times [JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m})] & \xrightarrow{\phi_n} & JX \\ \downarrow id \times \psi_n & \searrow \varphi & \\ f^*JH \times P(f) & & \end{array}$$

where φ denotes the addition map. According to Lemma 2.6, φ is an isogeny of degree $(16m)^{2g-2}$. Hence ψ_n is an isogeny of degree

$$\deg \psi_n = \frac{\deg \phi_n}{\deg \varphi} = \frac{m^{2g-2} \cdot 2^{[(2^{r+1}-r-1)m+r-1](g-1)}}{(2^r m)^{2g-2}} = 2^{[(2^{r+1}-r-1)m-(r+1)](g-1)}$$

\square

Proof of Theorem 4.1. The following diagram is commutative

$$\begin{array}{ccc} JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) & & \\ \downarrow \simeq & \searrow \psi_n & \\ [JX_{K_\tau} \times P(b_\tau)] \times [JX_{K_{\tau\sigma^m}} \times P(b_{\tau\sigma^m})] & & P(f) \\ \downarrow \varphi_1 \times \varphi_2 & \nearrow a & \\ JX_\tau \times JX_{\tau\sigma^m} & & \end{array}$$

where φ_1 and φ_2 denote the addition maps. According to Lemma 2.6, φ_1 and φ_2 are isogenies of degrees $2^{8m(g-1)+2-s_0}$ and $2^{8m(g-1)+2-s_1}$ respectively. This implies that a is an isogeny of degree

$$\deg a = \frac{\deg \psi_n}{\deg \varphi_1 \cdot \deg \varphi_2} = \frac{2^{[(2^{r+1}-r-1)m-(r+1)](g-1)}}{2^{(2^r m-2)(g-1)}} = 2^{[(2^r-r-1)m-(r-1)](g-1)}.$$

which completes the proof of the theorem. \square

REFERENCES

- [1] Ch. Birkenhake, H. Lange: *Complex Abelian Varieties*. Second edition, Grundlehren der Math. Wiss. 302, Springer - Verlag, 2004.
- [2] A. Carocca, S. Recillas, R. Rodriguez: *Dihedral groups acting on Jacobians*. Contemp. Math. **311** (2002), 41–77.
- [3] D. Mumford: *Prym varieties I*. Contributions to Analysis, L.V. Ahlfors, I. Kra, B. Maskit, and L. Nirenberg, editors, Academic Press, 1974, 325–350.
- [4] A. Ortega, *Variétés de Prym associées aux revêtements n -cycliques d’une courbe hyperelliptique*. Math. Z. **245** (2003), 97–103.
- [5] S. Recillas, R. Rodriguez: *Prym varieties and fourfold covers II, the dihedral case*. Contemp. Math. **397** (2006), 177–191.
- [6] J. Ries: *The Prym variety of a cyclic unramified cover of a hyperelliptic curve*. J. reine angew. Math. **340** (1983), 59–69.

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